

# AXLER-ZHENG TYPE THEOREM ON A CLASS OF DOMAINS IN $\mathbb{C}^n$

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**ABSTRACT.** We prove a version of Axler-Zheng's Theorem on smooth bounded pseudoconvex domains in  $\mathbb{C}^n$  on which the  $\bar{\partial}$ -Neumann operator is compact.

When studying operators on spaces of holomorphic spaces, one of the important questions is about the compactness of operators. It has been known that on Bergman spaces, it is the Berezin transform that plays the crucial role in characterizing compact operators. The well-known Axler-Zheng Theorem [AZ98] provides the characterization of compactness of finite sums of finite products of Toeplitz operators with  $L^\infty$  symbols on the unit open disk  $\mathbb{D}$ . This result has been completed by Suárez [Suá07] to include operators in the Toeplitz algebra even on the unit ball of  $\mathbb{C}^n$ . Engliš [Eng99] prove the analog of Axler-Zheng Theorem for Fock (the Segal-Bargmann) space and irreducible bounded symmetric domains in  $\mathbb{C}^n$ . We note that Axler-Zheng Theorem is also true on the polydisk as the irreducibility condition in [Eng99] can be removed (see [CKL09, pg. 232]. Raimondo [Rai00] found a proof of Axler-Zheng Theorem for different domains in  $\mathbb{C}$ . Recently, Mitkovski and Wick extended Axler-Zheng Theorem to the Bergman spaces  $A^p(\mathbb{D}^n)$  and with Suárez they treated weighted Bergman spaces on the unit ball in  $\mathbb{C}^n$  (see [MSW, MW]).

The purpose of this paper is to provide a version of Axler-Zheng Theorem for quite general domains in  $\mathbb{C}^n$ , namely for smooth bounded pseudoconvex domains in  $\mathbb{C}^n$ . This setting provides a framework for the use of the methods from several complex variables, in particular the  $\bar{\partial}$ -Neumann problem. However, that forces symbols to be continuous up to the boundary. Before, we state our main result, we will introduce notation and basic definitions.

Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ . The Bergman space  $A^2(\Omega)$  is the closed subspace of  $L^2(\Omega)$  (with Lebesgue measure  $dV$ ) consisting of holomorphic functions on  $\Omega$ . Let  $P : L^2(\Omega) \rightarrow A^2(\Omega)$  denote the Bergman projection. For  $\phi \in L^\infty(\Omega)$ , the Toeplitz operator with symbol  $\phi$  is defined as  $T_\phi f = P(\phi f)$ , for  $f \in A^2(\Omega)$ . Let  $K$  denote the Bergman kernel for  $\Omega$ . Then  $K$  is the unique function on  $\Omega \times \Omega$  with the following

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*Date:* April 29, 2013.

*2010 Mathematics Subject Classification.* Primary 47B35; Secondary 32A07.

*Key words and phrases.* Axler-Zheng theorem, Toeplitz operators, pseudoconvex domains.

property:  $f(z) = \langle f, K(z, \cdot) \rangle$ , for every  $f \in A^2(\Omega)$  and  $z \in \Omega$ . Then

$$k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}}$$

is called the normalized Bergman kernel. One can check that  $\|k_z\| = 1$ . Let  $\mathcal{B}(A^2(\Omega))$  denote the algebra of bounded linear operators on  $A^2(\Omega)$ . For  $S \in \mathcal{B}(A^2(\Omega))$ , the Berezin transform of  $S$  is defined as the function

$$\tilde{S}(z) = \langle Sk_z, k_z \rangle \text{ for } z \in \Omega.$$

We define the Berezin transform of a function  $\phi \in L^\infty(\Omega)$  as  $\tilde{\phi} = \tilde{T}_\phi$ .

Axler-Zheng Theorem states that if  $S$  is a finite sum of finite products of Toeplitz operators on  $A^2(\mathbb{D})$  with symbols in  $L^\infty(\mathbb{D})$ , then  $S$  is compact if and only if  $\tilde{S}(z) \rightarrow 0$  as  $|z| \rightarrow 1^-$ .

Our method uses standard arguments from the  $\bar{\partial}$ -Neumann problem. When the domain  $\Omega$  is bounded pseudoconvex, as a result of Hörmander's work, the complex Laplacian  $\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*$  (here  $\bar{\partial}^*$  is the Hilbert space adjoint of  $\bar{\partial}$ ) has a bounded inverse  $N$ , known as the  $\bar{\partial}$ -Neumann operator, on the square integrable  $(0, 1)$ -forms on  $\Omega$ . We refer the reader to [CS01, Str10] for more information on the  $\bar{\partial}$ -Neumann problem.

Let  $\mathcal{T}(\bar{\Omega})$  denote the norm closed subalgebra of  $\mathcal{B}(A^2(\Omega))$  generated by  $\{T_\phi : \phi \in C(\bar{\Omega})\}$ . This algebra has been studied by Coburn [Cob74] in the case of  $\Omega$  is the open unit disk in  $\mathbb{C}$  and by Axler, Conway, and McDonald [ACM82] in case of  $\Omega$  is a nonempty bounded domain in  $\mathbb{C}$ . Now we state our main result.

**Theorem 1.** *Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  on which  $\bar{\partial}$ -Neumann operator is compact. Assume that  $T \in \mathcal{T}(\bar{\Omega})$  and  $\lim_{z \rightarrow p} \tilde{T}(z) = 0$  for all  $p \in b\Omega$ . Then  $T$  is compact.*

*Remark 1.* The class of domains where the  $\bar{\partial}$ -Neumann operator is compact is very large; yet it excludes certain domains with analytic disks in the boundary. A set  $X$  is said to have an analytic disc if there exists a nonconstant holomorphic mapping  $f : \mathbb{D} \rightarrow X$ . The  $\bar{\partial}$ -Neumann operator is compact on strongly pseudoconvex domains and more generally on domains with property  $(P)$  (or  $(\tilde{P})$ ) (see [Cat84, McN02] and [Str10, Theorem 4.8]). For example, a bounded convex domain satisfies Property  $(P)$  if and only if there is no analytic disk in its boundary (see [FS98] and [Str10, Theorem 4.26]). Hence the next corollary follows immediately.

**Corollary 1.** *Let  $\Omega$  be a smooth bounded convex domain in  $\mathbb{C}^n$  whose boundary has no analytic disk. Assume that  $T \in \mathcal{T}(\bar{\Omega})$  and  $\lim_{z \rightarrow p} \tilde{T}(z) = 0$  for all  $p \in b\Omega$ . Then  $T$  is compact.*

*Remark 2.* As mentioned in Remark 1, Theorem 1 applies to a large class of domains in  $\mathbb{C}^n$ . However, it excludes some simple domains such as the polydisk as the  $\bar{\partial}$ -Neumann operator is not compact on the polydisk [Sal91] (see also [Kra88, FS98]). In case  $\Omega$  is a smooth bounded strongly pseudoconvex in  $\mathbb{C}^n$ , Theorem 1 is a special case of [AE01, Theorem 2.3].

**Corollary 2.** *Let  $\Omega$  be a smooth bounded convex domain in  $\mathbb{C}^2$  and  $\phi_j \in C^1(\bar{\Omega})$  for  $1 \leq j \leq m$ . Assume that  $\phi_j \circ f$  is holomorphic for all  $j$  and any holomorphic function  $f : \mathbb{D} \rightarrow b\Omega$ . Then the following are equivalent*

- i.  $T = T_{\phi_1} \cdots T_{\phi_m}$  is compact,
- ii.  $\lim_{z \rightarrow p} \tilde{T}(z) = 0$  for all  $p \in b\Omega$ ,
- iii. the product  $\phi_1 \phi_2 \cdots \phi_m = 0$  on  $b\Omega$ .

*Remark 3.* Corollary 2 applies to finite sums of finite products of Toeplitz operators as well. Let  $\Omega$  be a smooth bounded convex domain in  $\mathbb{C}^2$  and  $\phi_j \in C(\bar{\Omega})$  for  $1 \leq j \leq m$ . Assume that  $\phi_j \circ f$  is holomorphic for all  $j$  and any holomorphic function  $f : \mathbb{D} \rightarrow b\Omega$  and that  $T$  is a finite sum of finite products of Toeplitz operators  $T_{\phi_j}$ . Then compactness of  $T$  is equivalent to the continuous function  $\phi$  being zero on  $b\Omega$  where  $\phi$  is the corresponding finite sum of finite products of the symbols  $\phi_j$ .

The techniques in this paper also give us some results about the essential norms of Toeplitz operators with symbols continuous up to the boundary. Let  $T : X \rightarrow Y$  be a bounded linear operator between two Banach spaces  $X$  and  $Y$ . Then the essential norm  $\|T\|_e$  of  $T$  is defined as

$$\|T\|_e = \inf\{\|T - S\| : S : X \rightarrow Y \text{ is a compact linear operator}\}.$$

We remind the reader that for any  $\psi \in C(\bar{\Omega})$  its sup norm on the boundary is defined as

$$\|\psi\|_{L^\infty(b\Omega)} = \sup\{|\psi(z)| : z \in b\Omega\}.$$

For  $\Omega$  a bounded domain in  $\mathbb{C}$ , Axler, Conway, and McDonald [ACM82] obtained the essential spectrum and the essential norm of an arbitrary Toeplitz operator with a symbol in  $C(\bar{\Omega})$ . The following result is an extension of their result about the essential norm.

**Proposition 1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n$  and  $\psi \in C(\bar{\Omega})$ . Then  $\|T_\psi\|_e \leq \|\psi\|_{L^\infty(b\Omega)}$ . Furthermore, if  $\Omega$  is smooth bounded pseudoconvex in  $\mathbb{C}^n$  such that the set of strongly pseudoconvex points is dense in  $b\Omega$  then  $\|T_\psi\|_e = \|\psi\|_{L^\infty(b\Omega)}$ .*

We get the following immediate corollaries due to the following facts: compactness of the  $\bar{\partial}$ -Neumann operator implies that the set of strongly pseudoconvex points is dense in the boundary in relative topology [SS06, Corollary 1]; and in case of convex domains,

absence of analytic disks in the boundary is equivalent to compactness of the  $\bar{\partial}$ -Neumann operator [FS98, Theorem 1.1].

**Corollary 3.** *Let  $\Omega$  be smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  and  $\psi \in C(\bar{\Omega})$  such that the  $\bar{\partial}$ -Neumann operator is compact. Then  $\|T_\psi\|_e = \|\psi\|_{L^\infty(b\Omega)}$ .*

**Corollary 4.** *Let  $\Omega$  be smooth bounded convex in  $\mathbb{C}^n$  with no analytic disks in the boundary and  $\psi \in C(\bar{\Omega})$ . Then  $\|T_\psi\|_e = \|\psi\|_{L^\infty(b\Omega)}$ .*

## PROOFS

*Proof of Proposition 1.* Let us fix  $\alpha > \|\psi\|_{L^\infty(b\Omega)} \geq 0$ . We will show that there exists a compact operator  $S$  such that  $\|T_\psi - S\| < \alpha$ . Since  $b\Omega$  is compact and  $|\psi| < \alpha$  on  $b\Omega$  there exists an open neighborhood  $U$  of  $b\Omega$  such that  $|\psi| < \alpha$  on  $U \cap \Omega$ . Let us choose a smooth cut-off function  $\chi \in C_0^\infty(\Omega)$  such that  $0 \leq \chi \leq 1$ ,  $\chi = 1$  on a neighborhood of  $\Omega \setminus U$ , and  $\chi = 0$  on  $b\Omega$ . Then  $S = T_{\chi\psi}$  is a compact operator because  $\chi\psi = 0$  on  $b\Omega$ . Furthermore,

$$\|T_\psi\|_e \leq \|T_\psi - S\| \leq \sup\{|(1 - \chi(z))\psi(z)| : z \in \bar{\Omega}\} < \alpha.$$

Since  $\alpha$  is arbitrary we get  $\|T_\psi\|_e \leq \|\psi\|_{L^\infty(b\Omega)}$ .

To prove the second part assume that  $p \in b\Omega$  is a strongly pseudoconvex point and  $\|T_\psi\|_e = \alpha_0$ . Then for every  $\varepsilon > 0$  there exists a compact operator  $K$  such that  $\|T_\psi - K\| < \alpha + \varepsilon$ . Applying the Berezin transform we get  $|\langle (T_\psi - K)k_z, k_z \rangle| < \alpha + \varepsilon$  for all  $z \in \Omega$ . If we let  $z$  converge to  $p$ , compactness of  $K$  implies that  $\langle Kk_z, k_z \rangle$  converges to zero and since  $p$  is strongly pseudoconvex we have

$$|\psi(p)| = \lim_{z \rightarrow p} |\tilde{\psi}(z)| < \alpha_0 + \varepsilon.$$

Therefore, we showed that  $|\psi(z)| \leq \alpha_0$  for any strongly pseudoconvex point  $z \in b\Omega$ . Then continuity of  $\psi$  on  $b\Omega$  and the assumption that the set of strongly pseudoconvex points is dense in the boundary imply that  $\|\psi\|_{L^\infty(b\Omega)} \leq \|T_\psi\|_e$ .  $\square$

To prove Theorem 1 we use the following lemma as well as Proposition 1.

**Lemma 1.** *Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$ ,  $\phi \in C(\bar{\Omega})$ , and  $p \in b\Omega$  be a strongly pseudoconvex point. Then  $\lim_{z \rightarrow p} \tilde{\phi}(z) = \phi(p)$ .*

*Proof.* Without loss of generality we will assume that  $\phi$  is real valued. Let  $B(p, r)$  denote the open ball centered at  $p$  with radius  $r$ . Continuity of  $\phi$  on  $\bar{\Omega}$  implies that for every  $\varepsilon > 0$  there exists  $\delta_1 > 0$  such that  $|\phi(w) - \phi(p)| < \varepsilon$  for  $w \in \bar{\Omega} \cap B(p, \delta_1)$ . Smoothness of the domain implies that there exists  $0 < r < \delta_1$  so that every point in  $B(p, r) \cap b\Omega$  is strongly pseudoconvex. Then  $\Gamma_1 = b\Omega \setminus \overline{B(p, r)}$  and  $\Gamma_2 = b\Omega \cap B(p, r/2)$  are disjoint open subsets of

$b\Omega$ . Then [Bel86, Theorem 2] (see also [Boa87, Theorem 2]) implies that the Bergman kernel  $K$  extends smoothly to  $\Gamma_1 \times \Gamma_2$ . Since  $K(z, z) \rightarrow \infty$  as  $z \rightarrow p$  ([Pfl75], see also [JP93, Theorem 6.1.7]), the normalized Bergman kernel  $|k_z|^2$  converges uniformly to 0 on  $\overline{\Omega} \setminus B(p, \delta_1)$  as  $z \rightarrow p$ . Then there exists  $0 < \delta_2 < r/2$  such that  $z \in \overline{\Omega} \cap B(p, \delta_2)$  implies that  $|k_z(w)|^2 < \varepsilon$  for all  $w \in \overline{\Omega} \setminus B(p, \delta_1)$ . First we will estimate  $\tilde{\phi}(z)$  from above for  $z \in \overline{\Omega} \cap B(p, \delta_2)$ . Let  $M = \int_{\Omega} |\phi(w)| dV(w) + 1$ . Then

$$\begin{aligned}
\tilde{\phi}(z) &= \int_{\Omega} \phi(w) |k_z(w)|^2 dV(w) \\
&\leq \int_{\Omega \setminus B(p, \delta_1)} |\phi(w)| |k_z(w)|^2 dV(w) + (\phi(p) + \varepsilon) \int_{\Omega \cap B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\leq \varepsilon \int_{\Omega \setminus B(p, \delta_1)} |\phi(w)| dV(w) + (\phi(p) + \varepsilon) \int_{\Omega \cap B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\leq \varepsilon \int_{\Omega} |\phi(w)| dV(w) + (\phi(p) + \varepsilon) \left( 1 - \int_{\Omega \setminus B(p, \delta_1)} |k_z(w)|^2 dV(w) \right) \\
&\leq \phi(p) + \varepsilon M + |\phi(p)| \int_{\Omega \setminus B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\leq \phi(p) + \varepsilon(M + |\phi(p)| V(\Omega)).
\end{aligned}$$

In the last step we used that fact that  $\int_{\Omega \setminus B(p, \delta_1)} |k_z(w)|^2 dV(w) \leq \varepsilon V(\Omega)$ , where  $V(\Omega)$  denotes the volume of  $\Omega$ . Hence for  $z \in \overline{\Omega} \cap B(p, \delta_2)$  we have

$$(1) \quad \tilde{\phi}(z) - \phi(p) \leq \varepsilon(M + V(\Omega)|\phi(p)|).$$

Similarly we estimate  $\tilde{\phi}(z)$  from below as follows:

$$\begin{aligned}
\tilde{\phi}(z) &= \int_{\Omega} \phi(w) |k_z(w)|^2 dV(w) \\
&\geq - \int_{\Omega \setminus B(p, \delta_1)} |\phi(w)| |k_z(w)|^2 dV(w) + (\phi(p) - \varepsilon) \int_{\Omega \cap B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\geq -\varepsilon \int_{\Omega \setminus B(p, \delta_1)} |\phi(w)| dV(w) + (\phi(p) - \varepsilon) \int_{\Omega \cap B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\geq -\varepsilon \int_{\Omega} |\phi(w)| dV(w) + (\phi(p) - \varepsilon) \left( 1 - \int_{\Omega \setminus B(p, \delta_1)} |k_z(w)|^2 dV(w) \right) \\
&\geq \phi(p) - \varepsilon M - |\phi(p)| \int_{\Omega \setminus B(p, \delta_1)} |k_z(w)|^2 dV(w) \\
&\geq \phi(p) - \varepsilon(M + |\phi(p)| V(\Omega)).
\end{aligned}$$

Hence for  $z \in \overline{\Omega} \cap B(p, \delta_2)$  we have

$$(2) \quad \tilde{\phi}(z) - \phi(p) \geq -\varepsilon(M + V(\Omega)|\phi(p)|).$$

Combining (1) and (2) we showed that for  $\varepsilon > 0$  there exists  $\delta_2 > 0$  so that

$$-\varepsilon M \leq \tilde{\phi}(z) - \phi(p) \leq \varepsilon M$$

for  $z \in \overline{\Omega} \cap B(p, \delta_2)$ . Therefore,  $\lim_{z \rightarrow p} \tilde{\phi}(z) = \phi(p)$ .  $\square$

*Remark 4.* The Berezin transform of a function that is continuous up to the boundary is continuous on strongly pseudoconvex points in the boundary.

**Corollary 5.** *Let  $\Omega$  be a smooth bounded pseudoconvex domain in  $\mathbb{C}^n$  such that the set of strongly pseudoconvex points is dense in  $b\Omega$ . Assume that  $\phi \in C(\overline{\Omega})$  and  $\lim_{z \rightarrow p} \tilde{\phi}(z) = 0$  for all strongly pseudoconvex points  $p \in b\Omega$ . Then  $T_\phi$  is compact.*

In the proof of Theorem 1 we use Hankel operators. The Hankel operator  $H_\phi : A^2(\Omega) \rightarrow L^2(\Omega)$  with a symbol  $\phi \in L^\infty(\Omega)$  is defined by  $H_\phi(f) = (I - P)(\phi f)$ . Kohn's formula  $P = I - \bar{\partial}^* N \bar{\partial}$  implies that  $H_\phi(f) = \bar{\partial}^* N \bar{\partial}(\phi f)$  for  $f \in A^2(\Omega)$  and  $\phi \in C(\overline{\Omega})$ . The following standard fact will be useful: if  $N$  is a compact operator on  $\Omega$  then  $H_\phi$  is compact for all  $\phi \in C(\overline{\Omega})$  (see [Str10, Propositions 4.1 and 4.2]).

*Proof of Theorem 1.* First we will prove the theorem in case of  $T$  is a finite product of Toeplitz operators with symbols continuous on  $\overline{\Omega}$ . Assume that  $T = T_{\phi_m} T_{\phi_{m-1}} \cdots T_{\phi_1}$  such that  $\phi_j \in C(\overline{\Omega})$  for all  $j$  and  $\lim_{z \rightarrow p} \tilde{T}(z) = 0$  for all  $p \in b\Omega$ . Then  $\tilde{T}$  has a continuous extension on  $\overline{\Omega}$  and  $\tilde{T} = 0$  on  $b\Omega$ .

Using the formula  $H_{\psi_1}^* H_{\psi_2} = T_{\psi_1 \psi_2} - T_{\psi_1} T_{\psi_2}$  inductively one can show that

$$\begin{aligned} T_{\phi_m} T_{\phi_{m-1}} \cdots T_{\phi_2} T_{\phi_1} &= T_{\phi_m \phi_{m-1} \cdots \phi_2 \phi_1} - \left( T_{\phi_m} T_{\phi_{m-1}} \cdots T_{\phi_3} H_{\phi_2}^* H_{\phi_1} \right. \\ &\quad \left. + T_{\phi_m} T_{\phi_{m-1}} \cdots T_{\phi_4} H_{\phi_3}^* H_{\phi_2 \phi_1} + \cdots + H_{\phi_m}^* H_{\phi_{m-1} \cdots \phi_2 \phi_1} \right). \end{aligned} \quad (3)$$

Compactness of the  $\bar{\partial}$ -Neumann operator implies that Hankel operators with symbols continuous on the closure are compact [Str10, Proposition 4.1 and 4.2]. Therefore, we have

$$T_{\phi_m} T_{\phi_{m-1}} \cdots T_{\phi_1} = T_{\phi_m \phi_{m-1} \cdots \phi_1} + K$$

where  $K$  is a compact operator.

Let  $\phi = \phi_m \phi_{m-1} \cdots \phi_1$  on  $\overline{\Omega}$ . Then  $T = T_\phi + K$  and

$$\tilde{T} = \tilde{T}_\phi + \tilde{K}.$$

Since  $K$  is compact we have  $\tilde{K} = 0$  on  $b\Omega$ . Hence,  $\tilde{T}_\phi = 0$  on  $b\Omega$ . However, Lemma 1 implies that  $\phi = \tilde{\phi} = \tilde{T}_\phi = 0$  on strongly pseudoconvex points. Since  $\bar{\partial}$ -Neumann operator is compact, [ŠS06, Corollary 1] implies that the set of strongly pseudoconvex points is dense in

the boundary (in the relative topology). This fact together with the continuity of  $\phi$  implies that  $\phi = 0$  on  $b\Omega$ . Hence,  $T_\phi$  and in turn  $T$  are compact.

In case  $T$  is a finite sum of finite products of Toeplitz operators with symbols continuous on  $\overline{\Omega}$  equation (3) implies that there exists  $\phi \in C(\overline{\Omega})$  and a compact operator  $K$  such that  $T = T_\phi + K$ . Then by the arguments in the previous paragraph we conclude that  $\phi = 0$  on  $b\Omega$ . Hence, in this case too,  $T$  is compact.

Finally we will prove the Theorem 1 for  $T \in \mathcal{T}(\overline{\Omega})$ . Using (3) we can assume that for every  $\varepsilon > 0$  there exist  $\psi_\varepsilon \in C(\overline{\Omega})$  and a compact operator  $K_\varepsilon$  such that  $\|T - T_{\psi_\varepsilon} + K_\varepsilon\| < \varepsilon$ . Then for  $z \in \Omega$  we have

$$\begin{aligned} \left| \tilde{T}(z) - \tilde{T}_{\psi_\varepsilon}(z) + \tilde{K}_\varepsilon(z) \right| &= \left| \langle T(k_z) - T_{\psi_\varepsilon}(k_z) + K_\varepsilon(k_z), k_z \rangle \right| \\ &\leq \|T - T_{\psi_\varepsilon} + K_\varepsilon\| \\ &< \varepsilon \end{aligned}$$

Since we assumed that  $\lim_{z \rightarrow p} \tilde{T}(z) = 0$  for all  $p \in b\Omega$  and  $K_\varepsilon$  is compact, we have

$$\limsup_{z \rightarrow p} \left| \tilde{T}_{\psi_\varepsilon}(p) \right| < \varepsilon$$

for every  $p \in b\Omega$ . However, Lemma 1 implies that  $|\psi_\varepsilon(p)| = \lim_{z \rightarrow p} |\tilde{\psi}_\varepsilon(z)| < \varepsilon$  for any strongly pseudoconvex point  $p \in b\Omega$ . Then using continuity of  $\psi_\varepsilon$  on  $\overline{\Omega}$  together with the fact that strongly pseudoconvex points are dense in the boundary, we deduce that  $|\psi_\varepsilon| < \varepsilon$  on  $b\Omega$ . Proposition 1 implies that there exists a compact operator  $S_\varepsilon$  such that  $\|T_{\psi_\varepsilon} - S_\varepsilon\| < \varepsilon$ . Then

$$\|T - S_\varepsilon + K_\varepsilon\| \leq \|T - T_{\psi_\varepsilon} + K_\varepsilon\| + \|T_{\psi_\varepsilon} - S_\varepsilon\| < 2\varepsilon.$$

Therefore,  $T$  is in the operator norm closure of subspace of compact operators in the operators. Since this subspace is closed in the operator norm topology we conclude that  $T$  is compact.  $\square$

To prove Corollary 2 we will need the following result. We note that even though the result below was stated for symbols that are smooth up to the boundary, its proof goes through for symbols that are continuously differentiable on the closure of the domain.

**Theorem 2** ([ČŠ09, Theorem 3]). *Let  $\Omega$  be a smooth bounded convex domain in  $\mathbb{C}^2$  and  $\phi \in C^1(\overline{\Omega})$ . If  $\phi \circ f$  is holomorphic for any holomorphic  $f : \mathbb{D} \rightarrow b\Omega$ , then  $H_\phi$  is compact.*

*Proof of Corollary 2.* The implication i.  $\Rightarrow$  ii. is a result of compactness of  $T$  and the fact that  $k_z$  converges to zero weakly as  $z$  converges to  $b\Omega$ .

To prove ii.  $\Rightarrow$  iii. we use Theorem 2 to conclude that  $H_{\phi_j}$  is compact for every  $j$ . This fact together with (3) implies that  $T = T_\phi + K$  where  $\phi = \phi_1\phi_2 \cdots \phi_n \in C(\overline{\Omega})$  and  $K$  is a compact operator. Then ii. implies that  $\tilde{\phi}$  has a continuous extension to the boundary and  $\tilde{\phi} = 0$  on  $b\Omega$ . On a strongly pseudoconvex point  $p \in b\Omega$  we have

$$\phi(p) = \lim_{z \rightarrow p} \tilde{\phi}(z) = 0.$$

Assume that  $D$  is a nontrivial analytic disk in  $b\Omega$ . Then there exists a complex line  $L \subset \mathbb{C}^2$  such that  $\overline{D} = L \cap \overline{\Omega}$  [ČŠ09, Lemma 2] (see also [FS98]).

Now we will show that the boundary of  $D$  is in the closure of strongly pseudoconvex points in  $b\Omega$ . Assume that there exists a point  $p$  in the boundary of  $D$  that is not in the closure of strongly pseudoconvex points in  $b\Omega$ . Then there exists a nontrivial (affine) analytic disk  $\tilde{D} \subset b\Omega$  through  $p$  (see, for example, [Fre74, Theorem 1.1]). Since  $\Omega \subset \mathbb{C}^2$  is convex we conclude that  $\tilde{D} = \tilde{L} \cap \overline{\Omega}$  for a complex line  $\tilde{L}$ . However,  $p \in L \cap \tilde{L}$  and since  $\Omega$  is smooth  $L$  and  $\tilde{L}$  are perpendicular to the same normal vector at  $p$ . Therefore,  $L = \tilde{L}$  and  $D = \tilde{D}$ . This contradicts with the assumption that  $p$  is a boundary point of the disk  $D$ .

The assumption that  $\phi \circ f$  is holomorphic for any holomorphic function  $f : \mathbb{D} \rightarrow b\Omega$  implies that the restriction  $\phi|_D$  is holomorphic on  $D$  for any analytic disk  $D \subset b\Omega$ . Then the maximum modulus principle applied to  $\phi|_D$  yields that  $\phi = 0$  on  $D$ . Then continuity of  $\phi$  on  $\overline{\Omega}$  implies that  $\phi_1\phi_2 \cdots \phi_n = \phi = 0$  on  $b\Omega$ .

To prove iii.  $\Rightarrow$  i., again we write  $T = T_\phi + K$  where  $\phi = \phi_1\phi_2 \cdots \phi_n \in C(\overline{\Omega})$  and  $K$  is a compact operator. Then compactness of  $T$  follows from the fact that  $T_\phi$  is compact whenever  $\phi = 0$  on  $b\Omega$ .  $\square$

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